DFTT 8/95
hep-th/9501104
January 1995

The 2D effective field theory of interfaces derived from 3D field theory

Paolo Provero * and Stefano Vinti †

Dipartimento di Fisica Teorica
Università di Torino
Via P. Giuria 1
10125 Torino, Italia

Abstract

The one–loop determinant computed around the kink solution in the 3D ϕ^4 theory, in cylindrical geometry, allows one to obtain the partition function of the interface separating coexisting phases. The quantum fluctuations of the interface around its equilibrium position are described by a $c = 1$ two–dimensional conformal field theory, namely a 2D free massless scalar field living on the interface. In this way the capillary wave model conjecture for the interface free energy in its gaussian approximation is proved.

*e–mail: provero@ge.infn.it
†e–mail: vinti@to.infn.it

1 Introduction

The physics of interfaces separating coexisting phases in 3D systems is dominated by long–wavelength, low–energy fluctuations; it is therefore natural to describe the interface fluctuations in terms of a 2D effective theory. A common assumption is to take the interface free energy to be proportional to the area of the interface: this is the well known capillary wave model (CWM) which is believed to describe the interface physics [1, 2]. More recently, the CWM predictions have been made explicit one order beyond the gaussian approximation and verified by means of numerical simulations on 3D spin systems to high accuracy [3, 4].

Despite these results, the CWM is an *ad hoc* 2D effective theory: it is not known in general how to derive it from the original 3D hamiltonians, except in the zero–temperature limit (see *e.g.* [5] and references therein).

In this paper we provide an analytical derivation of the 2D effective theory of interfaces in the framework of 3D Euclidean ϕ^4 theory, which is known to describe the scaling region of the Ising model (for a general review see for instance [6]). Our result reproduces the predictions of the CWM in its gaussian approximation: the partition function of an interface is proportional to the partition function of a 2D, $c = 1$ conformal field theory (CFT), namely a free scalar field living on the interface.

This result can be thought of as a new instance of dimensional reduction: the relevant degrees of freedom of a physical system are described by an effective theory of lower dimensionality.

To be more precise, we consider the 3D ϕ^4 theory in a cylindrical geometry with two of the three space–time dimensions having finite lengths L_1, L_2 and periodic boundary conditions. Using ζ–function regularization, we compute, in one–loop approximation, the energy–gap $E(L_1, L_2)$ due to tunneling: in the dilute–gas approximation, this quantity is proportional to the partition function of an interface.

The paper is organized as follows: in Sec. 2 we establish our notations and review the expression of $E(L_1, L_2)$ in terms of a functional determinant, regularized using the ζ–function method. In Sec. 3 we evaluate the determinant: our main result is the expression (34) for $E(L_1, L_2)$. Sec. 4 is devoted to some concluding remarks.

2 The interface partition function

Consider the 3D field theory defined by the action

$$S[\phi] = \int d^3x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + V(\phi)\right] \tag{1}$$

where

$$V(\phi) = \frac{g}{4!}\left(\phi^2 - v^2\right)^2 \tag{2}$$

in Euclidean space–time with finite size in the "spatial" directions x_i ($i = 1, 2$) but infinite in the "time" direction x_0. We put periodic boundary conditions on the finite sizes:

$$\phi(x_0, x_1 + L_1, x_2) = \phi(x_0, x_1, x_2 + L_2) = \phi(x_0, x_1, x_2) \quad . \tag{3}$$

The potential V has two degenerate minima in $\phi = \pm v$ and a maximum in $\phi = 0$.

A solution of the equations of motion connecting the two minima is the kink

$$\phi_{cl}(x) = v \tanh\left[\frac{m}{2}(x_0 - a)\right] \tag{4}$$

where

$$m = \left(\frac{gv^2}{3}\right)^{1/2} \quad , \tag{5}$$

and its action is

$$S_c \equiv S[\phi_{cl}] = \frac{2m^3}{g}L_1L_2 \quad . \tag{6}$$

The existence, in finite volume, of classical solutions connecting the two degenerate minima of the potential, and hence of a non–vanishing tunneling probability between the two minima, has the effect of removing the double degeneracy of the vacuum which, in infinite volume, is due to the spontaneous breaking of the Z_2 symmetry $\phi \to -\phi$. The energy splitting is given, in one–loop approximation, by (see *e.g.* [7])

$$E(L_1, L_2) = 2e^{-S_c}\left(\frac{S_c}{2\pi}\right)^{1/2}\left|\frac{\det' M}{\det M_0}\right|^{-1/2} \tag{7}$$

where M is the operator

$$M = -\frac{\partial}{\partial x_\mu}\frac{\partial}{\partial x_\mu} + V''(\phi_{cl}(x)) \quad . \tag{8}$$

Here $\det'$ indicates the determinant without the zero mode, which is due to the freedom in choosing the kink location a, and gives rise, when treated with the collective coordinates method, to the prefactor $(S_c/2\pi)^{1/2}$. M_0 is the free–field fluctuation operator

$$M_0 = -\partial_\mu \partial_\mu + m^2 \quad . \tag{9}$$

The computation of the energy splitting (7) for the symmetric case $L_1 = L_2$ was done in Ref. [8]. We will see that the generalization of the calculation to asymmetric geometries allows one to recognize the interface partition function as the partition function of a $2D$ CFT.

We use ζ–function regularization to compute the ratio of determinants appearing in Eq.(7). It is useful to express the operators M and M_0 as

$$M = Q(x_0) - \partial_i \partial_i \qquad (i = 1, 2) \tag{10}$$

$$M_0 = Q_0(x_0) - \partial_i \partial_i \tag{11}$$

where

$$Q(x_0) = -\partial_0^2 + m^2 - \frac{3}{2} m^2 \frac{1}{\cosh^2\left[\frac{m}{2}(x_0 - a)\right]} \tag{12}$$

$$Q_0(x_0) = -\partial_0^2 + m^2 \quad . \tag{13}$$

The regularized ratio of determinants appearing in Eq. (7) is then expressed as

$$\frac{\det' M}{\det M_0} = \exp\left\{ -\frac{d}{ds}\left[\zeta_M(s) - \zeta_{M_0}(s)\right]\Big|_{s=0} \right\} \tag{14}$$

where the ζ–function of an operator A with eigenvalues a_n is defined as

$$\zeta_A(s) = \sum_n a_n^{-s} \tag{15}$$

The spectra of the operators Q, Q_0 and $-\partial_i \partial_i$ are known, and the relevant ζ–function is

$$\begin{aligned} \zeta_M(s) - \zeta_{M_0}(s) \quad &= \quad {\sum_{n_1,n_2}}' (\lambda_{n_1,n_2})^{-s} + \sum_{n_1,n_2} \left(\lambda_{n_1,n_2} + \frac{3}{4} m^2 \right)^{-s} \\ &+ \quad \sum_{n_1,n_2} \int_{-\infty}^{+\infty} dp \; g(p) \left(\lambda_{n_1,n_2} + p^2 + m^2 \right)^{-s} \end{aligned} \tag{16}$$

where the primed sum runs over $(n_1, n_2) \neq (0,0)$. Here $\lambda_{n_1 n_2}$ are the eigenvalues of the two–dimensional operator $-\partial_i \partial_i$ with periodic boundary conditions on the rectangle of sides L_1, L_2:

$$\lambda_{n_1 n_2} = 4\pi^2 \left(\frac{n_1^2}{L_1^2} + \frac{n_2^2}{L_2^2} \right) \qquad n_1, n_2 \in Z \quad . \tag{17}$$

$g(p)$ is the difference between the spectral densities of Q and Q_0:

$$g(p) = -\frac{m}{2\pi} \left(\frac{2}{p^2 + m^2} + \frac{1}{p^2 + \frac{m^2}{4}} \right) \quad . \tag{18}$$

3 Evaluation of the determinant

To complete our calculation we have to evaluate the ζ–function (16). Following Refs. [8, 9] we write

$$\zeta_M(s) - \zeta_{M_0}(s) \equiv \zeta_1(s) + \zeta_2(s) \tag{19}$$

where

$$\begin{aligned} \zeta_1(s) &= \sum_{n_1, n_2}{}' \lambda_{n_1, n_2}^{-s} & (20) \\ \zeta_2(s) &= \sum_{n_1, n_2} \left\{ \left(\lambda_{n_1 n_2} + \frac{3m^2}{4} \right)^{-s} \right. \\ &\quad \left. + \int_{-\infty}^{+\infty} dp \; g(p) \left(\lambda_{n_1 n_2} + p^2 + m^2 \right)^{-s} \right\} & (21) \end{aligned}$$

The term ζ_1 can be recognized to be the ζ–function of a massless, 2D free scalar field on the rectangle of sides L_1, L_2 with periodic boundary conditions, *i.e.* on a torus [10]. From 2D CFT we know that its derivative in $s = 0$ is [10]

$$\left. \frac{d\zeta_1}{ds} \right|_{s=0} = -2 \log \left[L_1 \left| \eta(\tau) \right|^2 \right] \tag{22}$$

where

$$\tau \equiv i \frac{L_1}{L_2} \tag{23}$$

is modular parameter of the torus and $\eta(\tau)$ is the Dedekind function. When combined with the prefactor $(S_c/2\pi)^{1/2}$ coming from the zero mode in Eq.

(7), this term produces precisely the modular invariant partition function of the $c = 1$ CFT defined by a free massless scalar field.

To evaluate $\zeta_2(s)$, we proceed like in Refs. [8, 9]: we write

$$\begin{aligned} \zeta_2(s) &= \frac{1}{\Gamma(s)} \sum_{n_1 n_2} \int_0^\infty dt\ t^{s-1} \left\{ \exp\left[-\left(\lambda_{n_1 n_2} + \frac{3m^2}{4} \right) t \right] \right. \\ & \left. + \int_{-\infty}^{+\infty} dp\ g(p) \exp\left[-\left(\lambda_{n_1 n_2} + p^2 + m^2 \right) t \right] \right\} \end{aligned} \tag{24}$$

and, introducing the Jacobi theta function

$$A(x) = \sum_n \exp\left(-\pi n^2 x \right) \quad , \tag{25}$$

we have

$$\zeta_2(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-1} A\left(\frac{4\pi t}{L_1^2} \right) A\left(\frac{4\pi t}{L_2^2} \right) F(m,t) \tag{26}$$

$$F(m,t) = \exp\left(-\frac{3}{4} m^2 t \right) + \int_{-\infty}^{+\infty} dp\ g(p) \exp\left[-\left(p^2 + m^2 \right) t \right] \quad . \tag{27}$$

Using Poisson's summation formula $A(x)$ is seen to satisfy

$$A(x) = x^{-1/2} A(1/x) \tag{28}$$

which we use to express ζ_2 as

$$\zeta_2(s) = \zeta_2^{(a)}(s) + \zeta_2^{(b)}(s) \tag{29}$$

where

$$\zeta_2^{(a)}(s) = \frac{L_1 L_2}{4\pi} \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-2} F(m,t) \tag{30}$$

$$\zeta_2^{(b)}(s) = \frac{L_1 L_2}{4\pi} \frac{1}{\Gamma(s)} \int_0^\infty dt\ t^{s-2} \left[A\left(\frac{L_1^2}{4\pi t} \right) A\left(\frac{L_2^2}{4\pi t} \right) - 1 \right] F(m,t) \tag{31}$$

The term $\zeta_2^{(b)}$ is exponentially suppressed for large L_1, L_2 [9] and will therefore be neglected in what follows. $\zeta_2^{(a)}$ is then computed straightforwardly:

$$\begin{aligned} \zeta_2^{(a)}(s) &= \frac{L_1 L_2}{4\pi} \frac{1}{s-1} m^{2(1-s)} \left\{ \left(\frac{3}{4} \right)^{(1-s)} - \frac{3}{2\pi} \frac{\Gamma(1/2)\Gamma(s-1/2)}{\Gamma(s)} \right. \\ & \left. - \frac{3}{8\pi} \int_{-\infty}^{+\infty} dq \frac{(q^2+1)^{-s}}{q^2 + 1/4} \right\} \end{aligned} \tag{32}$$

so that

$$\left.\frac{d\zeta_2^{(a)}}{ds}\right|_{s=0} = -\frac{3m^2 L_1 L_2}{4\pi}\left(1+\frac{1}{4}\log 3\right) \quad . \tag{33}$$

Therefore the $\zeta_2^{(a)}$ term provides simply a quantum correction to the interface tension.

Substituting (22) and (33) in (14) and (7) we finally obtain

$$E(L_1, L_2) = \frac{C}{[Im(\tau)]^{1/2}\,|\eta(\tau)|^2}\exp(-\sigma L_1 L_2) \tag{34}$$

where

$$C = \frac{2}{\sqrt{\pi}}\left(\frac{m^3}{g}\right)^{1/2} \tag{35}$$

$$\sigma = -\frac{2m^3}{g}\left[1+\frac{3g}{16\pi m}\left(1+\frac{1}{4}\log 3\right)\right] \quad . \tag{36}$$

In Ref. [8] the energy gap was computed in the symmetric case $L_1 = L_2$, in which the τ–dependent contribution reduces to a constant. Notice that in [8] the energy gap is expressed in terms of the physical mass m_{phys} (inverse of the correlation length) and the renormalized coupling $u_R \equiv g_R/m_R$ where the renormalized parameters g_R and m_R are defined according to a particular renormalization scheme. However it is important to keep in mind that, at one–loop, the renormalized parameters differ from the bare ones by *finite* quantities: the one–loop Feynman diagrams in 3D ϕ^4 are finite after dimensional continuation. The formulae needed to make contact between our result and the one quoted in Ref. [8] are

$$\frac{g}{m} \equiv u = u_R\left(1+\frac{31u_R}{128\pi}+\mathcal{O}(u_R^2)\right) \tag{37}$$

$$m^2 = m_{phys}^2\left[1+\frac{u_R}{16\pi}\left(-4+3\log 3\right)+\mathcal{O}(u_R^2)\right] \quad . \tag{38}$$

4 Conclusions

The effective, long–wavelength 2D theory of interface fluctuations in 3D ϕ^4 theory has been derived from first principles by analytical methods. The interface partition function turns out to be proportional to the partition function of a free massless 2D scalar field living on the interface. In this way

one is able to obtain a 2D conformal invariant field theory by dimensional reduction of 3D field theory.

This result is in agreement with the predictions of the capillary wave model of interfaces, which *assumes* an interface free energy proportional to the interface area. Indeed, the capillary wave model in its gaussian approximation predicts exactly the functional form (34) for the interface partition function [11, 3].

The predictions of the capillary wave model were tested against Monte Carlo simulations of spin systems in Refs. [3, 4]. In particular in [4] the model was successfully verified *beyond* the gaussian approximation: it would be interesting to investigate whether the CWM contributions beyond the gaussian one can be derived in a field–theoretic framework.

Acknowledgements

We are grateful to M. Caselle, F. Gliozzi and R. Levi for useful discussions.

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